59 Astigmatism Notation: Dealing with Vectors

Kristian Næser

Net astigmatisms and spherocylinders are complex formats with magnitudes in diopters (D) and directions in degrees (°). These entities represent individual keratometries or refractions, but must be converted to vectors to allow for calculations of surgically induced astigmatism (SIA), toric intraocular lens (IOL) power, averages, spreads, etc. Several methods for reporting surgically induced astigmatism (SIA) have been reported during the last 40 years [1–16]. However, this issue has remained contested up till this day due to the continued use of nonvector methods [17–23].

This chapter describes the principles for conversion of refractive data to dioptric vectors. Næser's polar value dioptric vectors are detailed [3, 7, 11, 16, 22], other dioptric vector formats are reviewed, their statistical assessments are described, a terminology is suggested, and practical calculations are demonstrated.

Spherocylinder Formats

Spheres and cylinders are lower-order (LO) aberrations, correctable with spectacle glasses or contact lenses. Higher-order aberrations are not part of this discussion. In an optimal spherical (*stigmatic* or point-like) optical system, a point in the object space is focused as a point image (Fig. 59.1) [16]. In a regular *astigmatic* optical system, an object point is focused as two mutually perpendicular line segments (Fig. 59.2) [16]. The optical effects of ocular astigmatism are blur in all fixation distances due to lack of point focus and distortion caused by unequal (differential) magnification of the retinal image in the various meridians. The blurs generated from a 1.0 D cylinder and a 0.5 D sphere are equivalent.

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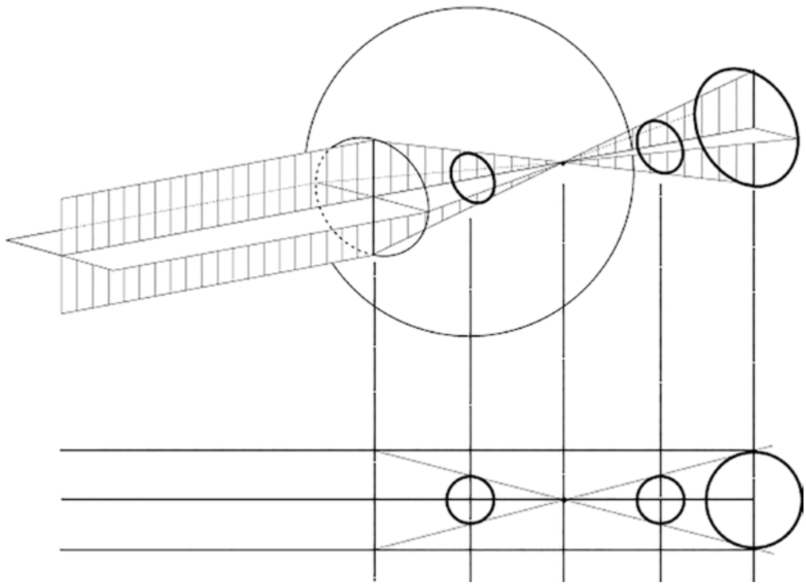

Fig. 59.1 A stigmatic ocular optical system. In the absence of diffraction, aberration, and scatter, a point in object space is focused as a point image. An object located in the far point is focused on the retina. Objects from any other position are defocused and blur circles are projected on the retina. (Reproduced with permission from John Wiley and Sons publishers)

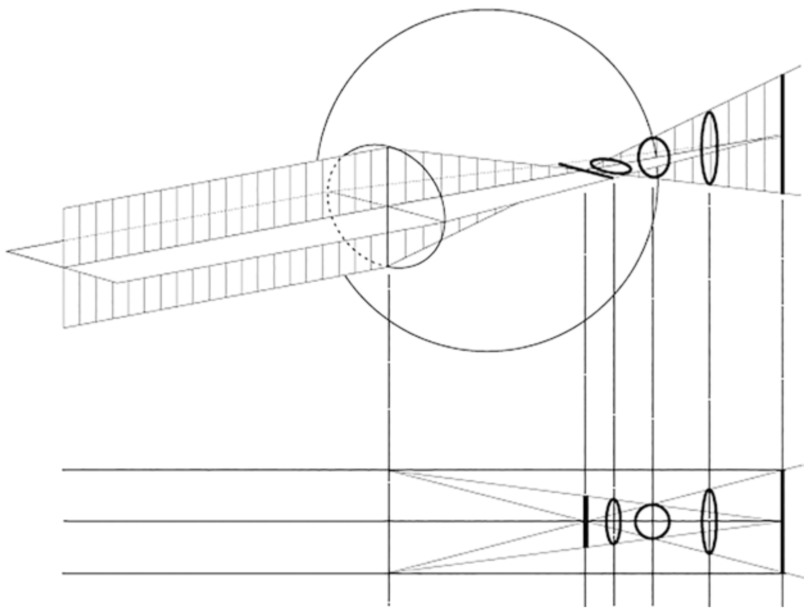

Fig. 59.2 An astigmatic with-the-rule ocular optical system. The mutually perpendicular focal lines delineate Sturm's interval. The position of the circle of least confusion is the dioptric average between the two focal lines and determines the spherical equivalent power. No point focus is formed. The image projected on the retina is always blurred due to its variable shapes and directions. (Reproduced with permission from John Wiley and Sons publishers)

Corneal Spherocylinder Formats

Keratometry, topography, and tomography should be based on independent measurements, employing stationary (not handheld) devices. A single autokeratometry is inadequate, but the vector average of three measurements is sufficient for precise assessment. (Fig. 59.3) [24–26].

Corneal measurements identify the radii of curvature ($R1$ and $R2$ in mm) and meridians along the two orthogonal principal meridians. The *curvature C* with units in *D* is the reciprocal of the radius of curvature [16]:

$$C = 1/R \qquad (59.1)$$

The radii of curvature in mm ($R1$ and $R2$) are converted to powers ($K1$ and $K2$) in diopters with the paraxial formula:

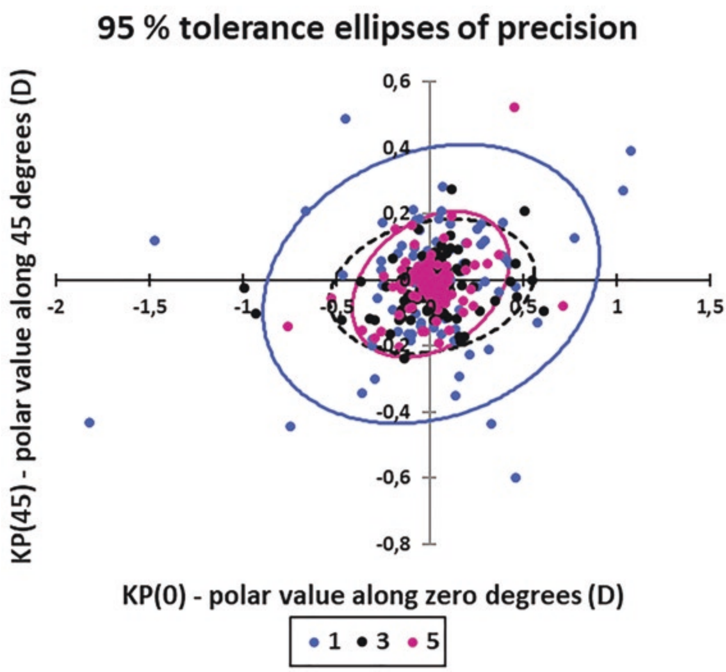

Fig. 59.3 Individual differences in astigmatism between paired Nidek Tonoref II autokeratometries with their 95% tolerance [24, 25] ellipses for observations in various measurement modalities. Bivariate plot. Units in diopters. Differences based on one measurement (blue) produced the largest (least precise) tolerance ellipse, while the ellipses for three (black) and five (violet) measurements were overlapping as a sign of approximately similar precision. Also note the reduction in outliers with increasing number of measurements

$$K = (n_2 - n_1)/R = \Delta n / R, \qquad (59.2)$$

where n_1 and n_2 are the refractive indices of the first and second medium along the optical pathway. The refractive index (n_1) of air = 1.0 is used in assessment of anterior corneal surfaces:

- *KA: Keratometric astigmatism* is based on measurement of the anterior surface only and employs an effective (or fictitious) refractive index n_2 of 1.3315 to 1.3375 to account for the average (but not measured) negative posterior corneal power (Fig. 59.4).
- *ACA: Anterior corneal astigmatism* also relies on measurement of the anterior surface and employs the actual refractive index of corneal tissue n_2 of 1.376.
- *PCA: Posterior corneal astigmatism* uses the corneal (n_1 = 1.376) and aqueous (n_2 = 1.336) refractive indices. The posterior corneal power is negative, because Δn (1.336–1.376 = −0.04) is negative (Fig. 59.5).
- *TCA: Total corneal astigmatism* employs tomographic measurements of both surfaces and of corneal thickness to allow thick lens optics or ray tracing with various methods. To date, such direct measurement of PCA has failed to outperform methods based on various mathematical modulations of KA magnitude and direction.

The spherical equivalent is the average of any orthogonal principal powers:

$$\mathrm{SE} = 0.5 \times (K1 + K2). \qquad (59.3)$$

The astigmatism magnitude M ($M \geq 0$) in diopters is the absolute difference between these two powers of maximal difference. For Næser's polar value system, a *plus power format* is selected. The astigmatic direction α in degrees is therefore defined as the *meridian* of *most positive* (or *least negative*) power. The net astigmatism M along the meridian α is symbolized as (M @ α). Corneal astigmatisms are subgrouped as with-the-rule (WTR), oblique, and against-the rule (ATR) according to the direction of the steep anterior meridian (Fig. 59.6).

Angular directions are given as values between zero and 180°. However, regular astigmatism is a

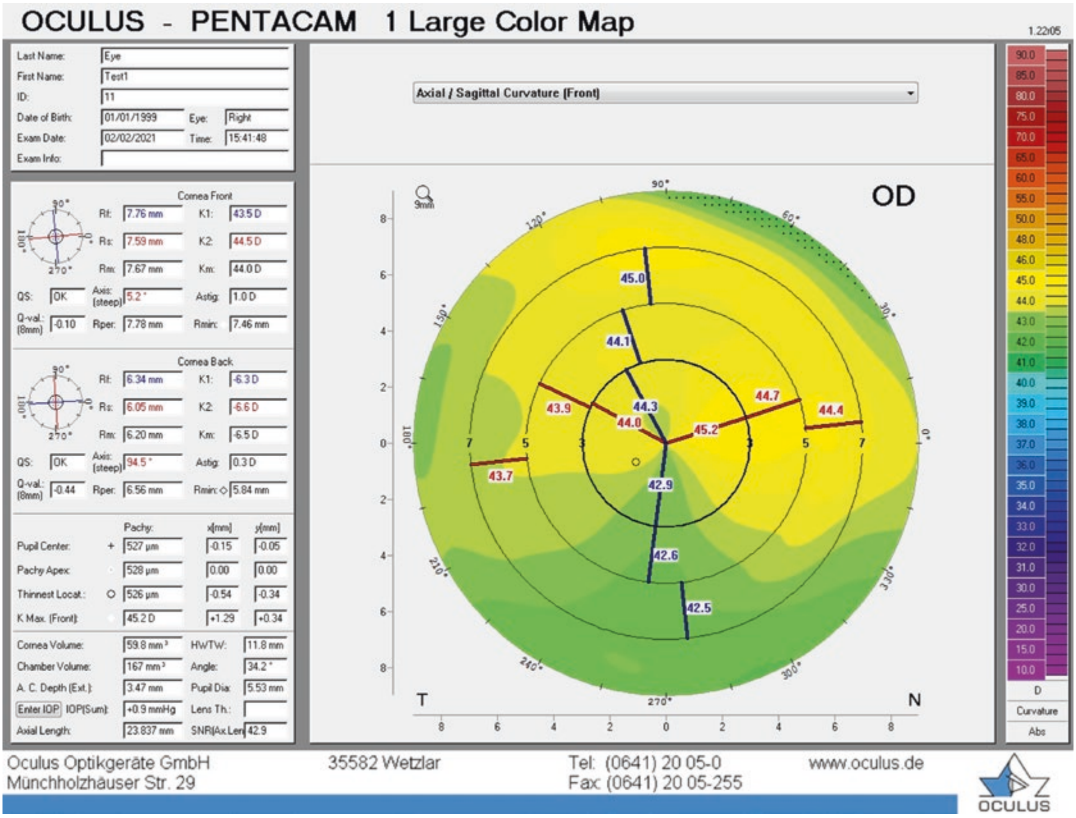

Fig. 59.4 Pentacam measurement of the anterior corneal surface. Simulated keratometry (top, left column) displays the steep meridian of (44.5 D @ 5.2°) and the flat meridian (43.5 D @ 95.2°). This is written in spherocylinder plus format as 43.5 D () 1.0 D @ 5.2°

periodic function, with a period of 180°. A direction of, for instance, 150° may be indicated as −30 or 330° for any practical and calculational purposes.

This correlation is described below, where p is an integer:

$$(M @ \alpha) = (M @ (\alpha + p \times 180)) \quad (59.4)$$

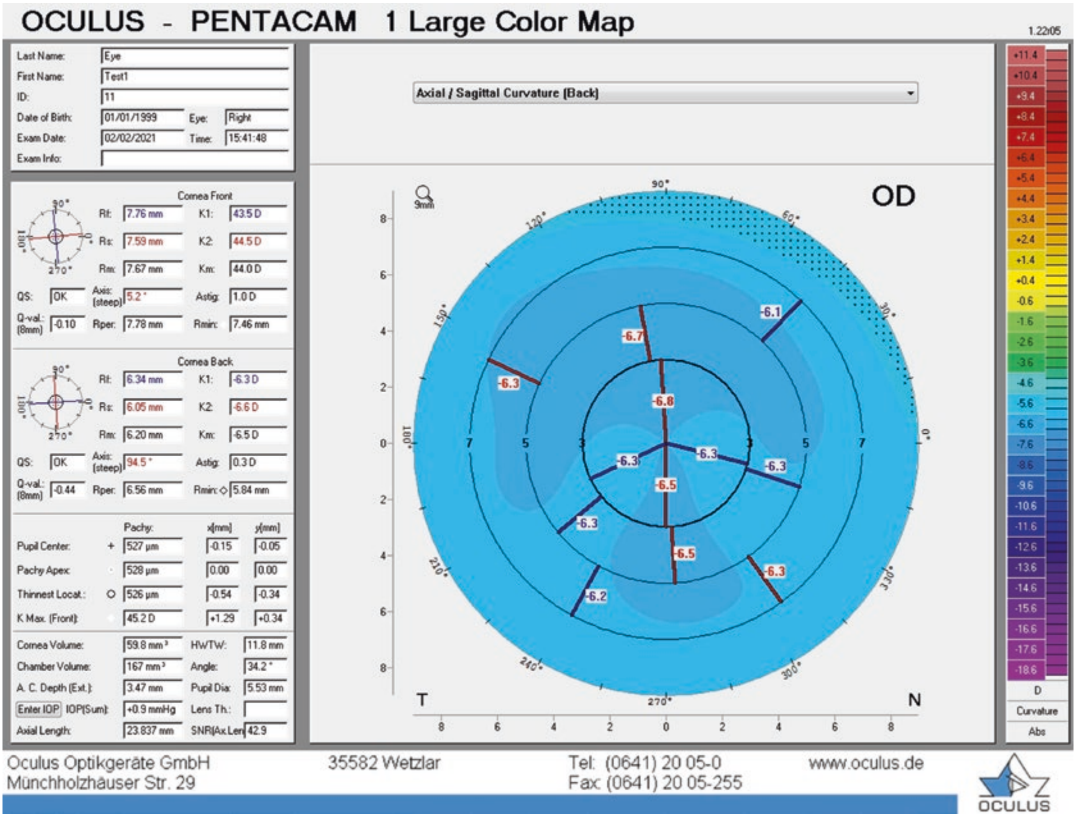

Fig. 59.5 Pentacam measurement of the posterior corneal surface (left column, middle) demonstrating a steep meridian of (−6.6 D @ 94.5°) and a flat meridian of (−6.3 D @ 4.5°). The spherocylinder plus format is (−6.6 D () 0.3 D @ 4.5°)

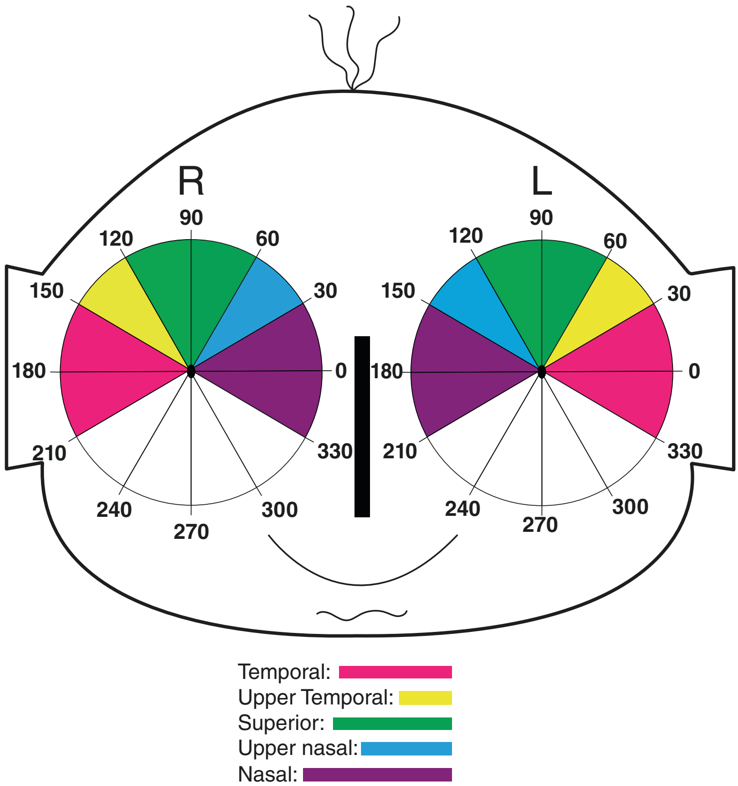

Fig. 59.6 The indication of corneal astigmatic direction is identical for right and left eyes as 0° towards the left ear, 90° superiorly, and 180° towards the right ear. Corneal astigmatisms are divided into subgroups as a function of the steep meridian direction; WTR (green), oblique (blue and yellow), and ATR (deep red and violet)

Refractive Spherocylinder Formats

Refraction is optimally based on autorefraction using a stationary (not handheld) device, but always finalized by a meticulous subjective (manifest) refraction in 0.25 D steps for sphere and cylinder, and in ≤5° steps for the axis. To compensate for differences in chart distance and thereby assure uniform reporting of distance refractions, measured refractions may be adjusted to infinity. The formula for this is:

$$\text{Refraction}_{\text{infinity}} = \text{Refraction}_{\text{actual}} - \left(1/\left(\text{chart distance in meters}\right)\right) \tag{59.5}$$

The spherocylinder prescription is given as: S () ($M \times \alpha$); where S is the sphere (D), M is the astigmatic magnitude (D), and x is the astigmatic axis (°). The SE is given as:

$$\text{SE} = S + 0.5 \times M \tag{59.6}$$

Both corneal and refractive spherocylinder data may be registered in a *plus* or *minus* cylinder format (Fig. 59.7). However, Næser's polar value system requires transformation to a plus format for both corneal and refractive data.

A *cross-cylinder* or *combined cylinder* format provides the cylinders along the orthogonal main axes. The cross-cylinders of the spherocylinder from Fig. 59.7 are written as (−4.75 D × 53°) and (−2,75 D × 143°). This format is used for conversion of each cross-cylinder from the vertex or spectacle plane (P_V) to the corneal surface (P_C), using the distance formula [9, 27]:

$$P_C = P_V / \left(1 - \left(P_V \times V / 1000\right)\right), \tag{59.7}$$

where V is the vertex distance in mm.

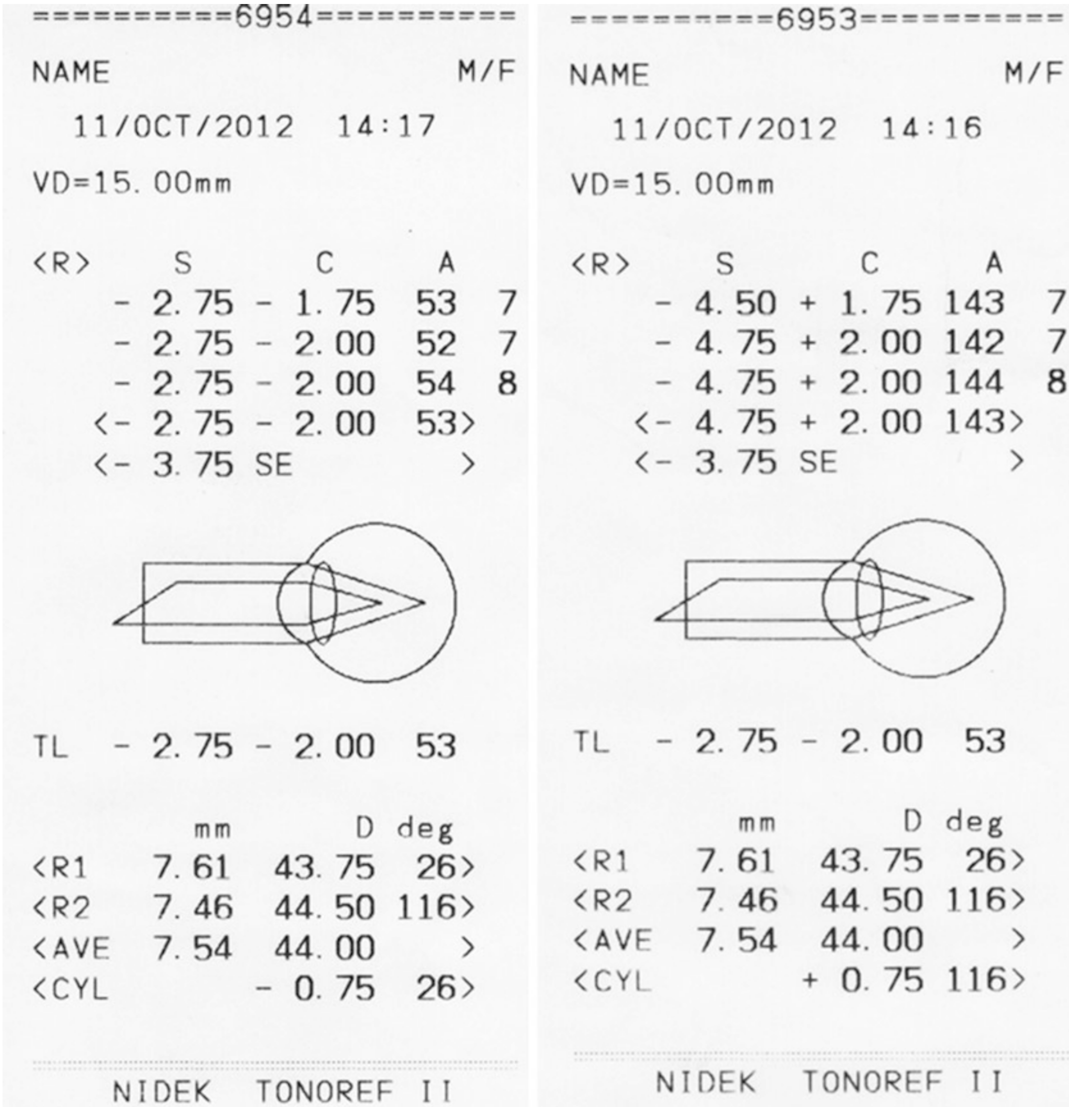

Fig. 59.7 Plus and minus formats for refractive and corneal measurements. Astigmatic directions are traditionally indicated by the *axes* for refractions and by the *meridians* for corneal measurements. The refractive spherical equivalent (SE) and the principle corneal meridians (here shown as *R*1 and *R*2) are always identical irrespective of the chosen plus or minus cylinder format

Cross cylinders are reconverted to the traditional plus or minus spherocylinder format in the corneal plane.

A similar *cross-cylinder* approach is employed for converting toric IOL toricity from the capsular to the anterior corneal plane.

Refractions and Aberrations

Consider a wavefront emanating from the eye. This is the total ocular aberration, generated by the ocular refractive surfaces and distances. The optical correction is the spherocylinder neutralizing the total ocular aberration. The refraction and the total ocular aberration at a given plane are therefore identical, but of opposite signs [16, 28]:

$$\text{Aberration} = -\text{Refraction} \tag{59.8}$$

This is well-known in clinical practise. A myopic eye is corrected with a minus sphere, because the refractive surfaces are too powerful relative to the vitreous length, hereby focusing parallel incident light in front of the retina. The meridian of a minus spectacle cylinder is aligned with the most powerful refractive meridian. Alternatively, correction may be conceived as a plus meridian cylinder aligned with the orthogonal less powerful meridian. As Næser's polar value system requires plus cylinder formats for calculations, the latter approach will be used in the following.

Comparing Corneal and Refractive Astigmatisms

These subtle correlations are not widely appreciated, but come out neatly [27, 28]. So, hang on!

Comparison of corneal and refractive data requires common plane, format, and angular definition. We choose the anterior corneal astigmatism as reference, here given as the net cylinder (M_C @ α_C).

The common plane is therefore the anterior corneal vertex, the format is aberration, and the direction is given by the plus power meridian.

Refractive data are transformed from vertex to corneal plane with Eq. (59.7) and subsequently reconverted to a plus axis format, symbolized as ($M_R \times \alpha_R$). The refraction is converted to aberration format by adding 90° to the direction (rather than chancing sign, hereby maintaining plus power format), as indicated in Eq. (59.8), yielding the net cylinder $M_R \times (\alpha_R + 90)$. Axis is changed to meridional direction by further addition (or subtraction) of 90°, changing the net cylinder to $M_R \times (\alpha_R + 180)$, which—according to Eq. (59.4)—is identical to ($M_R \times \alpha_R$). The entered refractive cylinder in axis format is therefore identical to the intended aberrational data in power format, and no further modifications are required for comparison! [27, 28]

Astigmatism directions are given as meridians for corneal measurements and as axes for refractions. Refractions are converted from the vertex to the anterior corneal plane to allow for comparison with corneal measurements.

Dioptric Vector Formats

Spherocylinders are converted to vectors in diopters by using meridional powers. Consider a spherocylinder in plus power format, S () (M @ α), with minimal power S along (α + 90)° and maximal power ($M + S$) along α. According to the sine-squared correlation, the meridional power along an oblique plane Φ is given as [16]:

$$\text{Meridional power along the plane}\,\Phi = S + M \times \cos^2(\alpha - \Phi) \tag{59.9}$$

Meridional powers along additional three planes are required (Figs. 59.8 and 59.9):

$$\text{Orthogonal plane along}(\Phi + 90) = S + M \times \cos^2(\alpha - (\Phi + 90)) = S + M \times \sin^2(\alpha - \Phi) \tag{59.10}$$

$$\text{Oblique plane along}(\Phi + 45), \text{counter} - \text{clockwise to}\,\Phi = S + M \times \cos^2(\alpha - (\Phi + 45)) \tag{59.11}$$

$$\text{Oblique plane along}(\Phi - 45), \text{clockwise to}\,\Phi = S + M \times \sin^2(\alpha - (\Phi + 45)) \tag{59.12}$$

All vector systems provide correct results, but vary with respect to reference meridians. Some systems use meridional powers directly, others

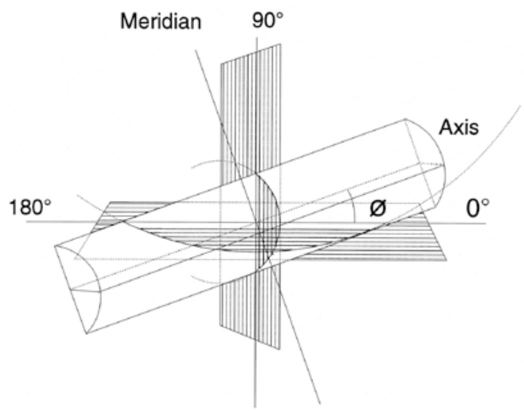

Fig. 59.8 A plano-cylinder in an oblique meridian. The meridional powers along 0 and 90° are illustrated by the radii of curvature in the hatched planes. KP(0), the polar value along the reference meridian Φ in 0°, is the dioptric difference between these meridional powers. (Reproduced with permission from Wolters Kluwer Health Inc.)

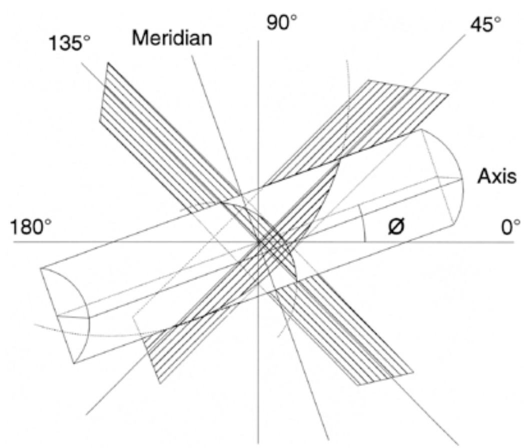

Fig. 59.9 The meridional powers along 45 and 135° are illustrated by the radii of curvature in the hatched planes. KP(45), the polar value along 45°, is the dioptric difference between these meridional powers. (Reproduced with permission from Wolters Kluwer Health Inc.)

rely on the difference between orthogonal meridional powers.

Næser's Polar Value System

The objective of refractive surgery is to flatten the steeper and/or to steepen the flatter corneal meridian or refractive wavefront, while simultaneously avoiding rotation of the cylinder direction. Næser's polar valuer system was designed for description of SIA by quantitating the flattening, steepening, and rotation of the surgical meridian.

A spherocylinder may be converted to a dioptric vector in the form of a SE power and two polar values, separated by an arch of 45° [3, 7, 11, 16, 22]. Optically, the polar values are cross cylinders in 45° inclinations. The three variables describe the spherocylinder completely, are mathematically independent, and allow for all types of algebraic operations.

The SE for corneal and refractive powers are reported in Eqs. (59.3) and (59.6). The astigmatism is completely characterized by the meridional and torsional polar values, with units in diopters [16]:

For the net cylinder (M @ α), the meridional polar value KP(Φ) along the (surgical) plane Φ is the difference in meridional powers along Φ and (Φ + 90), as reported in Eqs. (59.9) and (59.10) [16]:

$$\text{Meridional polar value} = M \times \cos\left(2 \times \left(\alpha - \Phi\right)\right) \tag{59.13}$$

The torsional polar value is the difference in meridional powers along the planes (Φ + 45) and (Φ − 45)), given in Eqs. (59.11) and (59.12):

$$\text{Torsional polar value}\,\text{KP}\left(\Phi + 45\right) = M \times \sin\left(2 \times \left(\alpha - \Phi\right)\right) = 2M \times \sin\left(\alpha - \Phi\right) \times \cos\left(\alpha - \Phi\right) \tag{59.14}$$

The magnitude of the torsional relative to the meridional power determines the angle and direction of cylinder relative to the reference plane Φ.

The polar value system always reports the change in astigmatism along the chosen meridian. This plane is usually the surgical meridian. The investigator chooses a *variable* or *fixed* reference plane Φ, for analysis of various clinical situations.

1. A *variable* reference plane along the preoperatively measured most powerful direction

for each eye is most useful for analysis of SIA. This direction varies for each eye, and Φ = preoperative value for α. *All* changes in astigmatism are referred to the preoperatively measured highest power of either the meridian (for cornea) or axis (for refraction). In this way, one can easily ascertain the refractive result, whether perfect or under- or overcorrected. For a preoperative astigmatic magnitude M and a target postoperative astigmatism of zero, the surgically induced meridional and torsional powers should be exactly $-M$ (for elimination of M) and zero (securing no axis rotation), respectively. For the SE and the meridional power, a power reduction or an overcorrection is indicated by a negative value, while an increase in power/under correction is given by a positive value. For the torsional power, a positive value indicates a counterclockwise and a negative value clockwise rotation of the cylinder meridian.

Recommended reference meridians for standard surgery:

Corneal incisional surgery: Variable reference meridians along the preoperative steep corneal meridians for on-median corneal incisional or fixed meridians (usually zero or 90°) for standard incisions.

Toric IOLs: The toric IOL axis = the predicted postoperative steep TCA meridian.

Corneal laser surgery: The plane of the preoperative refractive cylinder in plus format.

2. A *fixed* value for the reference plane of Φ = zero is used for analysis of temporal corneal incisions placed exactly in 0°. It may also be employed for interim calculations and population statistics within a with-the-rule (WTR) and against-the-rule (ATR) concept. By inserting Φ = zero in Eqs. (59.13) and (59.14), we obtain:

$$\mathrm{KP}(0) = M \times \cos(2 \times \alpha) \tag{59.15}$$

$$\mathrm{KP}(45) = M \times \sin(2 \times \alpha) \tag{59.16}$$

KP(0) is positive for ATR astigmatism and negative for WTR astigmatism. KP(45) is positive for oblique angular direction from 1 to 89°, and negative in angular directions between 90 and 180°.

3. Analysis of SIA following any other *fixed* direction is constructed similarly. For instance, in a right-handed surgeon consistently using a main corneal phacoemulsification incision in 100°, meridional and torsional powers emerge as:

$$\mathrm{KP}(100) = M \times \cos(2 \times (\alpha - 100)) \tag{59.17}$$

$$\mathrm{KP}(145) = M \times \sin(2 \times (\alpha - 100)) \tag{59.18}$$

The dynamics of SIA are best understood by remaining in the vector space and conceptually visualizing any surgically induced change in the terms of variation in SE and the meridional and torsional vectors. However, any single astigmatism and any compilation of astigmatisms may be converted to traditional cylinder format with the following equations: [16]

The astigmatic magnitude M in diopters:

$$M = \sqrt{\mathrm{KP}(\Phi)^2 + \mathrm{KP}(\Phi + 45)^2} \tag{59.19}$$

The astigmatic direction α in degrees:

$$\alpha = \arctan\left(\frac{M - \mathrm{KP}(\Phi)}{\mathrm{KP}(\Phi + 45)}\right) \tag{59.20}$$

The direction is given relative to the chosen reference meridian.

Other Vector Formats

Most dioptric power formats employ a SE power, combined with two cross cylinders in fixed directions along 0 and 45°, as described in Eqs. (59.15) and (59.16) [4, 6, 9, 10, 13, 14]. This analysis is correct within a WTR/ATR context, but cannot generally be interpreted as over- or under-correction for any surgical meridian. Signs may vary due to different definitions of direction. Thibos'

J0 and J45 use the half value of the astigmatic magnitude [14]. The equations reported by Holladay [4] for with-the-wound (incision) and Against-the-wound change are identical to the meridional powers in Eqs. (59.9) and (59.10).

Long's power matrix [29] relies on meridional powers with *axes* along zero and 90° (Eqs. 59.9 and 59.10) together with a torsional component (Eq. 59.16) and takes the following general formats:

$$f_{11} = S + M \times \sin^2(\alpha); f_{22} = S + M \times \cos^2(\alpha); f_{21} = f_{12} = -M \times \sin(\alpha) \times \cos(\alpha) \quad (59.21)$$

This vector format was used by Kaye [12] and Harris [15] for analysis of SIA.

Statistical Analysis

Descriptive and analytical statistical analyses of univariate SE, meridional, and torsional powers are performed with Excel or other database facilities [22]. Bivariate [24, 25] (simultaneous) statistical analysis of meridional and torsional powers may require data transfer to designated statistical programs. In double-angled plots, the "centroid" is the combined, bivariate mean of the individual distributions, while the extension of the confidence ellipse reflects the variability [16, 21, 24, 25]. *Accurate* surgical procedures are characterized by average differences in refraction close to zero. Univariate average means are assessed with a Student's t-test, bivariate means with Hotelling's T [24, 25]. Univariate and bivariate averages of multiple procedures are compared with analysis of variance (ANOVA) and multiple analysis of variance (MANOVA), respectively. *Precise* procedures have small standard deviations and variances, narrow confidence limits, and small confidence ellipses [24–26]. Bivariate analyses are based on the averages and standard deviations of and the correlation between *both* polar values and are therefore more reliable than univariate assessments. The total variance (TV) for an astigmatism entity is the sum of the meridional and torsion variances [24, 25]. The total standard deviation, TSD, is the square root of TV. Two variances may be compared with an F-test and multiple variances with Levene's or similar homogeneity test. All procedures mentioned assume normally distributed data.

Univariate non-normal data may be assessed with nonparametric statistics.

The mean absolute error (MAE) for astigmatism is the average of individually calculated magnitudes without consideration of directions, calculated with Eq. (59.19). The MAE relies on both the bias from the average error and from the variability of the individual observations. After conversion to this reduced scalar, directional data regarding over- or under-correction are irretrievably lost.

Terminology

The clinician *measures* the *preoperative* and a *postoperative* corneal and refractive power. The clinician *chooses* the *target refraction*. All other results are derived from these variables.

The terminology is identical for corneal and refractive measurements and include the Preoperative, Postoperative, Target, Surgically Induced, Target Induced, and the Error in *spherical equivalent* and *astigmatism*. Each astigmatism is further characterized by its meridional and torsional polar values. Abbreviations are listed in Table 59.1. The **SISE** and **SIA** are calculated as the **vector** differences between the postoperative and the preoperative measurements. The Target Induced Spherical Equivalent (**TISE**) and astigmatism (**TIA**) are defined as the **vector** differences between the target and the preoperative variables. The errors in spherical equivalent (**EISE**) and astigmatism (**EIA**) are calculated as the **vector** differences between the postoperative and the target values [22].

Dioptric power vectors are based on meridional powers. All vector systems provide correct calculations. They differ by the choice of reference meridians and definitions of direction. Some systems use meridional powers

Table 59.1 (Næser). Terminology with abbreviations [22]. All units in diopters (D). The astigmatism is further fully characterized by its meridional and torsional polar values. (Reproduced with permission from Wolters Kluwer Health Inc.)

Terminology	Spherical equivalent	Astigmatism
Preoperative	PRESE	PREA
Postoperative	POSE	POA
Target	TSE	TA
Surgically induced	SISE	SIA
Target induced	TISE	TIA
Error in	EISE	EIA

directly, others rely on the difference between orthogonal meridional powers.

Practical Calculation of Surgically Induced Refractive Change

Clinicians may copy the described formulas for their own use. A number of commercial and free web-based programs for calculating corneal and refractive changes are available. See for example: https://www.researchgate.net/publication/324106602_Surgically_induced_astigmatism_SIA_calculator_using_dr_Naesers_polar_value_system_Version_10_Free_software http://links.lww.com/JRS/A281.

Abulafia and Koch's double-angled plots for corneal and refractive SIA are freely available on the American Society of Cataract and Refractive Surgery and the Journal of Cataract and Refractive Surgery homepages.

Calculating the Corneal SIA for a Temporal Incision

The error in astigmatism (**EIA**) for toric IOL calculation is the vector difference between the corneal plane postoperative refractive astigmatism (**POA**) and the calculated target astigmatism (**TA**). The **TA** is the vector sum of the preoperative total corneal astigmatism (**TCA**), corneal **SIA**, and corneal plane **IOL toricity**. Vector calculation of corneal **SIA** is demonstrated in the following as an example.

We performed Nidek Tonoref II autokeratometry once before and 6 weeks after 2.4 mm clear corneal phakoemulcification incisions in 99 right eyes. All incisions were placed temporally in 180°. For SIA analysis, we therefore chose a fixed reference meridian in zero (=180) degrees. Autokeratometries were converted to SE power and the meridional (KP(0) and torsional (KP(45) polar values. The surgically induced change was calculated as the vector differences between the preoperative and postoperative values.

Data are summarized in Tables 59.2 and 59.3 and in Fig. 59.10. The three variables were normally distributed with a Kolmogorov-Smirnov test. The corneal incisions induced a statistically significant 0.09 D flattening and a nonsignificant 0.06 D counterclockwise torsion. The bivariate mean ("centroid") differed significantly from zero (Hotelling's T^2 = 0.008). Both average surgically induced polar values should be used for

Table 59.2 Key values for univariate description of the surgically induced refractive change. **SISE** is the surgically induced spherical equivalent power. The deviation of the average values from zero was tested with a two-tailed, univariate t-test (right column)

Univariate data		
Spherical equivalent (SE) power	Mean (SD); min–max value (D)	t-test
SISE	0.07 (0.30); −1.26–1.04	0.03
ASTIGMATISM (SIA)		
Meridional power (KP (0))	−0.09 (0.44); −1.36–1.52	0.03
Torsional power (KP(45))	−0.06 (0.32); −0.93–0.91	0.06

Table 59.3 Metrics for bivariate SIA analysis. Some metrics may be recognized from Fig. 59.10. The total variance (TV) is the sum of meridional and torsional univariate variances. The TSD is the square root of TV

SIA bivariate data	
"Centroid" as combined mean polar values (D)	(−0.09, −0.06)
"Centroid" as net astigmatism	0.11 D @ 107°
Pearson correlation	−0.19
Hotelling T^2 p-value	0.008
Total variance (TV)	0.29
Total standard deviation (TSD)	0.54
Mean absolute error (MAE)	0.46 D

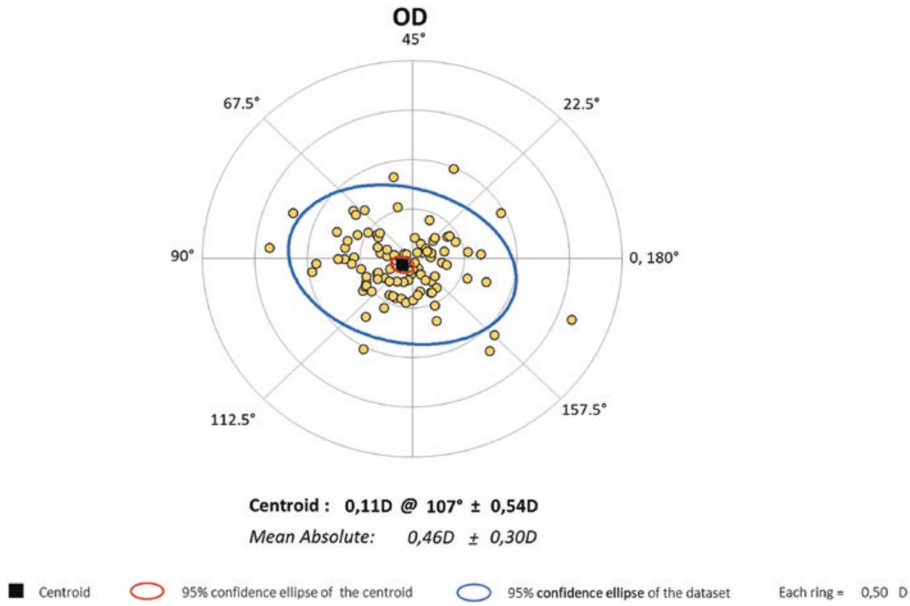

Fig. 59.10 Individual measurements and confidence ellipses for corneal SIA following temporal 2.4 mm phakoemulcification incisions in 99 right eyes. Each ring represents 0.5 D. Abscissa: meridional polar value KP(90. Ordinate: torsional polar value KP(45). Large black dot represents the bivariate mean ("centroid") in (−0.09 D, −0.06 D) = the net astigmatism (0.11 D @ 107°). Red line: 95% confidence ellipse of the mean. The bivariate mean differed statistically significantly from zero, as the origin (0,0) was outside this confidence ellipse. Blue line: 95% confidence ellipse for observations (also called tolerance ellipse). The two ellipses are congruent, and their relative axes length is the square root of the number of observations, or approximately ten in this example

future toric IOL calculations with similar corneal incisions. However, in contemporary web-based toric IOL calculation, only the flattening effect is considered. This is not entirely correct, but in this case a flattening of 0.1 D for "SIA" should be used for enhanced future average accuracy.

The spread is considerable in Fig. 59.10. Is this spread caused by the surgery or by the corneal measurements? The confidence ellipses for observations in Fig. 59.3 were constructed on a different data set of non-operated eyes, but with the same autokeratometer. The standard deviations for the paired difference between two single autokeratometries (blue confidence ellipse) amounted to 0.37 and 0.17 for KP(0) and KP(45). An F-test comparison of variances between Fig. 59.3 (blue ellipse) and the surgical case in Fig. 59.10 revealed a statistically significant ($p < 0.001$) difference for KP(45), but no ($p = 0.10$) difference for KP(0). A proportion of the spread in Fig. 59.10 may therefore be caused by the measurements. In Fig. 59.3, the variances based on three to five measurements were reduced to a third. Using the vector average of at least three autokeratometries will tend to enhance the precision of toric IOL calculation.

After transformation of corneal and refractive data to dioptric vectors, all calculations and statistical analyses may be performed in a univariate or a bivariate manner.

References

1. Jaffe NS, Clayman HM. The pathophysiology of corneal astigmatism after cataract extraction. Trans Am Acad Ophthalmol Otolaryngol. 1975;79:615–30.
2. Cravy TV. Calculation of the change in corneal astigmatism following cataract extraction. Ophthalmic Surg. 1979;10:38–49.
3. Naeser K. Conversion of keratometer readings to polar values. J Cataract Refract Surg. 1990;16:741–5.
4. Holladay JT, Cravy TV, Koch DD. Calculating the surgically induced refractive change following ocular surgery. J Cataract Refract Surg. 1992;18:429–43.

5. Kaye SB, Campbell SH, Davey K, Patterson A. A method for assessing the accuracy of surgical technique in the correction of astigmatism. Br J Ophthalmol. 1992;76:738–40.
6. Alpins NA. A new method of analyzing vectors for changes in astigmatism. J Cataract Refract Surg. 1993;19:524–33.
7. Naeser K, Behrens JK, Næser EV. Quantitative assessment of corneal astigmatic surgery: expanding the polar values concept. J Cataract Refract Surg. 1994;20:162–8.
8. Olsen T, Dam-Johansen M. Evaluating surgically induced astigmatism. J Cataract Refract Surg. 1994;20:517–22.
9. Holladay JT, Dudeja DR, Koch DD. Evaluating and reporting astigmatism for individual and aggregate data. J Cataract Refract Surg. 1998;24:57–65.
10. Alpins N. Astigmatism analysis by the Alpins method. J Cataract Refract Surg. 2001;27:31–49.
11. Naeser K, Hjortdal J. Polar value analysis of refractive data. J Cataract Refract Surg. 2001;27:86–94.
12. Kaye SB, Patterson A. Analyzing refractive changes after anterior segment surgery. J Cataract Refract Surg. 2001;27:50–60.
13. Holladay JT, Moran JR, Kezirian GM. Analysis of aggregate surgically induced refractive change, prediction error, and intraocular astigmatism. J Cataract Refract Surg. 2001;27:61–79.
14. Thibos LN, Horner D. Power vector analysis of the optical outcome of refractive surgery. J Cataract Refract Surg. 2001;27:80–5.
15. Harris WF. Analysis of astigmatism in anterior segment surgery. J Cataract Refract Surg. 2001;27:107–28.
16. Næser K. Assessment and statistics of surgically induced an astigmatism. Acta Ophthalmol. 2008 (Issue Thesis 1);86:1–28.
17. Reinstein DZ, Archer TJ, Randleman JB. JRS standard for reporting astigmatism outcomes of refractive surgery. J Refract Surg. 2014;30:654–9.
18. Næser K. Surgically induced astigmatism: distinguishing between vectors and non-vectors. J Refract Surg. 2015;31:349–50.
19. Næser K. Surgically induced astigmatism is characterized by optical vectors, not by ratios. J Cataract Refract Surg. 2016;43:347–8.
20. Reinstein DZ, Archer TJ, Srinivasan S, Mamalis N, Kohnen T, Dupps WJ Jr, Randleman JB. Standard for reporting refractive outcomes of intraocular lens-based refractive surgery. J Cataract Refract Surg. 2017;43:435–9.
21. Abulafia A, Koch DD, Holladay JT, Wang L, Hill W. Pursuing perfection in intraocular lens calculations. IV. Rethinking astigmatism analysis for intraocular lens-based surgery: suggested terminology, analysis, and standards for outcome. J Cataract Refract Surg. 2018;44:1169–74.
22. Næser K. Surgically induced astigmatism made easy: calculating the surgically induced change in sphere and cylinder for corneal incisional, corneal laser, and intraocular lens–based surgery. J Cataract Refract Surgery. 2021;47:118–22.
23. Koch DD, Wang L, Abulafia A, Holladay JT, Hill W. Rethinking the optimal methods for vector analysis of astigmatism. J Cataract Refract Surgery. 2021;47:100–5.
24. Naeser K, Hjortdal J. Bivariate analysis of surgically induced regular astigmatism. Mathematical analysis and graphical display. Ophthal Physiol Opt. 1999;19:50–61.
25. Naeser K. Hjortdal: multivariate analysis of refractive data. Mathematics and statistics of spherocylinders. J Cataract Refract Surg. 2001;27:129–42.
26. Javadi-Ottosen S, Næser K. Precision of the Nidek Tonoref II autokeratometer: how many repeated measurements are required? Acta Ophthalmol. 2021;99:611–5.
27. Bregnhøj JF, Mataji P, Næser K. Refractive, anterior corneal and internal astigmatism in the pseudophakic eye. Acta Ophthalmol. 2015;93:33–40.
28. Næser K. Combining refractive and topographic data in corneal refractive surgery for astigmatism. A new method based on polar value analysis and mathematical optimization. Acta Ophthalmol. 2012;90:768–72.
29. Long WF. A matrix formalism for decentration problems. Am J Optom Physiol Optic. 1976;53:27–33.